Some possible applications of Game Theory in some social dynamics

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These notes report a summary of our graduate thesis on some possible applications of Game Theory in some social dynamics by the following scheme:

Chapter 0: Some elements of Game Theory

Chapter 1: Social coordination

Chapter 2: Tax evasion

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Chapter 0: Some elements of Game Theory

A game is a situation in which individuals (*players*) can choose respectively certain behaviors (*strategies*), with the aim to maximize respectively their benefits (*payoffs*), taking into account that their respective payoffs depend on the strategies chosen by all players.

The goal of Game Theory – for short *TG* – is to predict, given a game, the behavior of the players and therefore the game outcome.

It is assumed that: (i) players interact consciously, i.e., players know that their payoff depends on the strategies chosen by all players, and this is a common knowledge ...; (ii) players are rational, i.e., players choose the best strategies to pursue their aim, and this is a common knowledge ...

The *normal-form representation* of a game specifies:
1) the set of *players*, say $\{1, ..., n\}$;
2) for $i = 1, ..., n$, the set of *strategies* at disposal to player i, say S_i;
3) for $i = 1, ..., n$, the *payoffs* function of player i, say $u_i : S_1\times\ldots\times S_n \to \mathbb{R}$.
This game is indicated as $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$.

A game is *static* if it takes place in a single phase [in which players choose a strategy simultaneously, or more generally without knowing the strategy chosen by the other players]. A game is *dynamic* if it takes place in several phases.

A game is with *complete information* if all players know the payoffs of all players for each combination of strategies. A game is with *incomplete information* if this information is only partial. In the sequel to this section we will cover only static games with complete information.

Nash Equilibrium (NE)

Nash's approach states that players' behavior is to converge on a game outcome called Nash Equilibrium and denoted below as *NE*.
The definition of NE is motivated as follows: if the task of Game Theory is to predict, given a game, the game outcome, then such an outcome must be "stable", i.e., players have no convenience to deviate [autonomously] from that outcome once it is verified, and "credible", i.e., players are willing to converge towards that outcome.

Definition (pure strategies)
In the game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$with n players, a *pure strategy* of player i (for $i = 1, ..., n$) is an element of the set S_i, i.e., it is a strategy at his disposal. □

Definition (NE in pure strategies)
In the game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$ with n players, strategies $(s_1^*, \ldots, s_n^*) \in S_1\times\ldots\times S_n$ are a *Nash Equilibrium in pure strategies of G* if, for each player i with $i = 1, ..., n$, one has:

$$u_i(s_1^*, \ldots, s_{i-1}^*, s_i^*, s_{i+1}^*, \ldots, s_n^*) \geq u_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*) \quad \text{for every } s_i \in S_i .$$

Player's i optimal response to strategies $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$, specified for the other n - 1 players, is the strategy i that solves the problem max $\{u_i(s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n): s_i \in S_i\}$

Then, in other words, the strategies $(s_1^*, \ldots, s_n^*) \in S_1 \times \ldots \times S_n$ are a Nash Equilibrium in pure strategies of G if each of them is an optimal response to the other ones. □

Then for a game G three cases may occur.

Case 1: G has 0 NE in pure strategies – for example the *Matching Pennies* game.

I \ II	Heads	Tails
Heads	– 1 , 1	1 , – 1
Tails	1 , – 1	– 1 , 1

In this case G has 0 NE in pure strategies.

Case 2: G has 1 NE in pure strategies – for example the *Prisoner's Dilemma* game.

I \ II	Stay silent	Betrays
Stay silent	– 1 , – 1	– 9 , 0
Betrays	0 , – 9	– 6 , – 6

In this case G has 1 NE in pure strategies: (Betrays, Betrays).

Case 3: G has multiple NE's in pure strategies – for example the *Battle of the Sexes* game.

I \ II	Football	Opera
Football	2 , 1	0 , 0
Opera	0 , 0	1 , 2

In this case G has 2 NE's in pure strategies: (Football, Football), (Opera, Opera).

Then one has that:

:: Case 1 provides a lack of prediction of the game outcome: this problem was studied by the same Nash through the introduction of mixed strategies and of Nash's Theorem;

:: Case 2 provides an exact prediction of the game outcome;

:: Case 3 provides both an indecision on predicting the game outcome and the possibility that players converge on towards a game outcome which is not a NE in pure strategies and – in particular – that they will get a payoff less than the payoff they would get by converging on a NE in pure strategies: such inconveniences were studied by Aumann through the introduction of Aumann's Correlated Equilibrium.

Case 1: *G* has 0 NE in pure strategies

For completeness we report in this section some concepts and results related to Case 1.

Let us recall that a *probability distribution* on a set $A = \{a_1, \dots a_n\}$ is an assignment $p = (p_1, \dots p_n)$ of values p_i over a_i, for $i = 1, ..., n$, such that $0 \leq p_i \leq 1$ and $p_1 + ... + p_n = 1$.

The following definitions are restricted to a game $G = \{S_1, S_2; u_1, u_2\}$ with 2 players according to Gibbons [2].

Definition (mixed strategies)
In a game $G = \{S_1, S_2; u_1, u_2\}$ with 2 players, a *mixed strategy* of player i (for i = I, II) is a probability distribution on S_i. □

Note that pure strategy is a special case of mixed strategy: in fact, any pure strategy can be expressed in terms of mixed strategy by a *probability distribution* in which one value is equal to 1 [the value corresponding to the pure strategy] and the other values equal to 0.

The concept of mixed strategy formalizes the possible indecision of a player about the strategy to choose, in the sense that for a player a mixed strategy is equivalent to randomly choose a strategy over the set of strategies at disposal to the player, on the basis of such a mixed strategy [i.e. of such a probability distribution]. Through the "criterion of expected value", see e.g. Gibbons [2], it is possible to calculate the payoff that players get. This makes the following definition possible.

Definition (NE in mixed strategies)
In a game $G = \{S_1, S_2; u_1, u_2\}$ with 2 players, the mixed strategies (p^*_1, p^*_2), one of player I, one of player II, are a *Nash Equilibrium in mixed strategies of G* if each of them is an optimal response to the other. □

Observation
In a game $G = \{S_1, S_2; u_1, u_2\}$with 2 players, each NE of G *induces* a probability distribution on $S_1 \times S_2$ [which is the set of possible outcomes of G] which assigns to every possible outcome of G the probability associated to that possible outcome provided that players behave according to that NE: that probability is obtained by multiplying, for every outcome, the probabilities associated to players' strategies which correspond to that outcome.

For example the *Battle of the Sexes* game (cf. Case 3) admits three NE's each one inducing a probability distribution on $S_1 \times S_2$ as follows:

		II	
		Football	Opera
	Football	$\underline{2}$, $\underline{1}$	0 , 0
I	Opera	0 , 0	$\underline{1}$, $\underline{2}$

:: the first NE is (Football, Football), in pure strategies, that is ((1, 0), (1, 0)) in terms of mixed strategies: it induces the following probability distribution on $S_1 \times S_2$

I/II	Lotta **(1)**	Opera **(0)**
Lotta **(1)**	1 · 1 = **1**	1 · 0 = **0**
Opera **(0)**	0 · 1 = **0**	0 · 0 = **0**

:: the second NE is (Opera, Opera), in pure strategies, that is ((0, 1), (0, 1)) in terms of mixed strategies: it induces the following probability distribution on $S_1 \times S_2$

I/II	Lotta **(0)**	Opera **(1)**
Lotta **(0)**	0 · 0 = **0**	0 · 1 = **0**
Opera **(1)**	1 · 0 = **0**	1 · 1 = **1**

:: the third NE is ((2/3, 1/3), (1/3, 2/3)), in mixed strategies: it induces the following probability distribution on $S_1 \times S_2$

I/II	Lotta **(1/3)**	Opera **(2/3)**
Lotta **(2/3)**	2/3 · 1/3 = **2/9**	2/3 · 2/3 = **4/9**
Opera **(1/3)**	1/3 · 1/3 = **1/9**	1/3 · 2/3 = **2/9**

Thus, according to the definition of mixed strategy, a NE in mixed strategies of a game can be seen as a [prediction of the players behavior and a] prediction of the game outcome in probabilistic terms. □

<u>Theorem (Nash, 1950)</u>
In a game $G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$(represented in normal-form) with n players, if n is finite and if S_i is finite for every i, then G has at least one Nash Equilibrium, possibly in mixed strategies. □

Then Nash's Theorem states that for each finite game [i.e., with a finite number of players, each with a finite number of strategies at disposal] there is at least one prediction of the game outcome in terms of NE, possibly in terms of NE in mixed strategies – that is in probabilistic terms.

Then, <u>for each finite game that satisfies Case 1</u>, even if there is no prediction of the game outcome in terms of NE in pure strategies, there is a prediction of the game outcome in terms of NE in mixed strategies – that is in probabilistic terms.

Case 3: *G* has multiple NE's in pure strategies

Recall the inconveniences of Case 3:

1. an indecision on predicting the game outcome;

2. the possibility that players converge on a game outcome which is not a NE in pure strategies and – in particular – that they get a payoff less than the payoff that they would get by converging on a NE in pure strategies.

Consider for example the *Battle of the sexes* game.

		II: Football	II: Opera
	Football	2 , 1	0 , 0
I	Opera	0 , 0	1 , 2

As already highlighted, the *Battle of the sexes* game has three NE's:

:: the first EN is (Football, Football), in pure strategies, i.e. ((1, 0), (1, 0)).
:: the second EN is (Opera, Opera), in pure strategies, i.e. ((0, 1), (0, 1)).
:: the third EN is in mixed strategies, i.e. ((2/3, 1/3), (1/3, 2/3)).

The inconvenience (1) is directly linked to the fact that the game has multiple EN's.
The inconvenience (2) is related to the fact that the third EN induces a probability distribution on $S_1 \times S_2$ (shown below) such that the probability that players converge on an outcome of G which is not a NE in pure strategies [that is converge on (Football, Opera) or on (Opera, Football)] is 4/9 + 1/9 = 5/9, and that in particular players would get a payoff (equal to 0) which is less than the payoff that they would get by converging on a NE in pure strategies.

I/II	Lotta **(1/3)**	Opera **(2/3)**
Lotta **(2/3)**	2/3 · 1/3 = **2/9**	2/3 · 2/3 = **4/9**
Opera **(1/3)**	1/3 · 1/3 = **1/9**	1/3 · 2/3 = **2/9**

Aumann's Correlated Equilibrium (simplified version) [1]

Let us report: (i) the definition of Aumann's Correlated Equilibrium in a simplified version, in particular restricted to a game with 2 players, but of course generalizable, and (ii) a scheme for its application in order to overcome the inconveniences (1) and (2).

Definition (Correlated Equilibrium)
In a game $G = \{S_1, S_2; u_1, u_2\}$ with 2 players, such that G satisfies Case 3, a Correlated Equilibrium of G is a probability distribution $p = (p_1, p_2, \ldots)$ on $S_1 \times S_2$ [which is the set of possible outcomes of G] such that $p_h = 0$ at each outcome h of G which is not a NE in pure strategies of G.□

Let $G = \{S_1, S_2; u_1, u_2\}$ be a game, with 2 players, such that G satisfies Case 3.

EC Scheme

Step 1: calculation of the EN's of G.

Step 2: the two players nominate a Mediator who: (i) randomly draws one of the possible outcomes of G on the basis of a probability distribution p on $S_1 \times S_2$ [which is the set of possible outcomes of G] which it is a Correlated Equilibrium of G (possibly proposed by the players); (ii) recommends to each player a strategy on the basis of the drawn outcome of G. □

NOTE: Since p is a Correlated Equilibrium of G, in Step 2-(i) the outcome of G drawn by the Mediator is a NE in pure strategies of G, and consequently in Step 2-(ii) to each player it is better to follow the Mediator's recommendation, assuming that the other player follows the Mediator's recommendation. So players finally will follow the Mediator's recommendations, and the outcome of G will be that drawn by the Mediator, i.e., it will be a NE in pure strategies of G.

For example let $G = \{S_1, S_2; u_1, u_2\}$ be the *Battle of the sexes* game.

		II: Football	II: Opera
I	Football	2 , 1	0 , 0
I	Opera	0 , 0	1 , 2

Game G satisfies Case 3 since it has multiple NE's.

Then let us apply the EC Scheme.

Step 1: the NE's in pure strategies of *G* are (Football, Football) and (Opera, Opera).

Step 2: players nominate a Mediator (which for example can be a third person).

In Step 2-(i) Mediator randomly draws one of the possible outcomes of *G* on the basis of a probability distribution *p* which is a Correlated Equilibrium of *G* [then Mediator draws a NE in pure strategies of *G*, i.e., either (Football, Football) or (Opera, Opera)].

By definition of Correlated Equilibrium, with reference to the table below, a probability distribution $p = (p_1, p_2, p_3, p_4)$ on $S_1 \times S_2$ [which is the set of possible outcomes of *G*] is a Correlated Equilibrium of *G* if $p_2 = 0$ and $p_3 = 0$.

I/II	Lotta	Opera
Lotta	p_1	p_2
Opera	p_3	p_4

In particular, the following probability distributions are Correlated Equilibria of *G*:
$(p_1, \ldots, p_4) = (1, 0, 0, 0)$
$(p_1, \ldots, p_4) = (0, 0, 0, 1)$
$(p_1, \ldots, p_4) = (1/2, 0, 0, 1/2)$
$(p_1, \ldots, p_4) = (1/3, 0, 0, 2/3)$
…

In Step 2- (ii) Mediator recommends to each player a strategy on the basis of the drawn outcome of *G*: for example, if the drawn outcome of *G* is (Opera, Opera), then Mediator recommends to player I strategy Opera and to player II strategy Opera. □

References

[1] R. Aumann, Correlated Equilibrium as an expression of Bayesian Rationality, *Econometrica* 55 (1987) 1-18

[2] R. Gibbons, Game Theory for Applied Economists, Princeton (1992)

[3] M. Li Calzi, Aumann e la Teoria dei Giochi, online

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Chapter 1: Social coordination

The main references of this chapter are those of Chapter 0 and the graduate theses of Marco di Giovanni [4] and of Matteo Paolini [6].

In this chapter: (i) we report some possible applications of Aumann's Correlated Equilibrium and (ii) we focus on one of such possible applications.

Concerning (i): with reference to Chapter 0, such possible applications are situations formalized through games that enjoy Case 3 [i.e., games that have multiple NE's in pure strategies] so that one may apply the EC Scheme. The aspect they have in common is the wish of *coordination* among players. Let us report such possible applications with the purpose to provide an input for the introduction of further possible applications.

Concerning (ii): we focus on one of such possible applications – that of the doctor – in order to calculate in detail the "advantage" that players would get through *coordination*. In particular it seems that the single "advantage" (of each player) does not depend on the number of players: this can be useful for applications in the field of social welfare, where the number of players is high, thus obtaining a sensible total "advantage".

1.1 Some possible applications of Aumann's Correlated Equilibrium

Let us report such possible applications by means of the following examples.

Example 1: traffic light (see [5])

Two drivers, say I and II, are located at a crossroads. Each of them can choose either to transit (T) or not to transit (NT). Let us represent this situation by a game G = {S_1, S_2; u_1, u_2} static with complete information, with 2 players, as represented below.

		II: T	II: NT
I	T	- 100 , - 100	1 , 0
I	NT	0 , 1	0 , 0

The NE's in pure strategies of G are (T, NT) and (NT, T).

For this game, players may nominate as a Mediator for example a traffic light that will randomly draw either (T, NT) or (NT, T) and will recommend to each player to choose either T or NT on the basis of the drawn outcome.

Note: This possible application is well known "a posteriori". □

Example 2: doctor (cf. [a1, a3])

Some patients must go to the doctor, without particular urgency, and must choose a day for that. In general it would be desirable a coordination among patients, so to overlap as less as possible their own choices, in order to minimize the waiting time.
Let us represent below a minimal instance of this situation by a game $G = \{S_1, S_2; u_1, u_2\}$ static with complete information, with 2 players, with S_1 = {M, T} and S_2 = {M, T} where M denotes Monday and T denotes Tuesday, and with payoffs which represent the estimated waiting times [that will be indicated by negative numbers since by convention the players have the aim to maximize their own payoff].

		II: M	II: T
I	M	-10 , -10	0 , 0
I	T	0 , 0	-10 , -10

The NE's in pure strategies of G are (M, T) and (M, T).

For this game, players may nominate as a Mediator for example the doctor, who will randomly draw either (L, M) or (M,L) and will recommend to each patient to choose either M or T on the basis of the drawn outcome.

In general for a real application: the doctor may establish a time interval of k days, then divide the patients into k balanced groups, and finally recommend to each group to choose a different day of the time interval. □

Example 3: leaving for holidays

Some people of a big city must leave for summer holidays, without particular urgency, and must choose the hourly band for that. In general it would be desirable a coordination among people, so to overlap as less as possible their own choices, in order to minimize [the traffic and so] the travel time.
Let us represent below a minimal instance of this situation by a game $G = \{S_1, S_2; u_1, u_2\}$ static with complete information, with 2 players, with S_1 = {A, B} and S_2 = {A, B} where A indicates a hourly band and B indicates another hourly band, and with payoffs which indicate the estimated journey time [that will be indicated by negative numbers since by convention the players have the aim to maximize their own payoff].

		II: A	II: B
I	A	-200 , -200	- 160 , -170
I	B	- 170 , -160	-210 , -210

The NE’s in pure strategies of *G* are (A, B), and (B, A).

For this game, players can nominate as a Mediator for example the Mayor that will randomly draw either (A, B) or (B, A), and will recommend to each player to choose either A or B on the basis of the drawn outcome.

In general for a real application: the Mayor could establish a range of *k* hourly bands, then divide people in *k* balanced groups balanced [for example on the basis of the plaques], and finally recommend to each group to choose a different band of the range. □

1.2 The possible application of the doctor of Example 2

In this section let us focus on the possible application of the doctor of Example 2, by starting from the minimal case, and then by gradually get to the general case. The context is that in which patients should go to the doctor, without particular urgency, and must choose a day for that. The idea of Aumann in this context is that it would be desirable a coordination among patients – so to overlap as less as possible their own choices – in order to minimize their waiting time.

1.2.1 The case with 2 patients and with 2 days

Let us represent the minimal case, with 2 patients and with 2 days, by the following game *G* static with complete information.

Game *G*:

Players: {Patient I, Patient II}

Strategies for each player: {M, T}

M = go to the doctor on Monday,

T = go to the doctor on Tuesday.

Payoffs: First let us introduce a table which is not that of payoff: the following table indicates, for each combination of strategies, the number of patients that any patient will find at the doctor [included himself].

Patient I , Patient II	M	T
M	2 , 2	1 , 1
T	1 , 1	2 , 2

For example, if Patient I chooses T and Patient II choses M, then: Patient I will find 1 patient at the doctor (i.e., himself), Patient II will find 1 patient at the doctor (i.e., himself).

Then let us denote as:
p = the number of patients who go to the doctor in a fixed day;

t = the average time for a doctor's appointment.
Then the (expected) *waiting time* of any patient that will go to the doctor in that fixed day is

$$(p - 1)\, t / 2 \qquad (0)$$

Proof of (0):
As p is the number of patients that go to the doctor in a fixed day, the number of patients that a Patient can already find at the doctor in that certain day, including the Patient himself, is a number $r \in \{1, \ldots, p\}$: thus the (expected) waiting time of the Patient is equal to $(r - 1)\, t$. On the other hand, assuming that all patients go randomly to the doctor, the probability that a Patient already finds at the doctor a number r of patients, including the Patient himself, is equal to $1/p$ for each $r \in \{1, \ldots, p\}$. It follows that the (expected) waiting time of the Patient is equal to

$$(1/p) \cdot 0 \cdot t + (1/p) \cdot 1 \cdot t + \ldots + (1/p)(p-1)\, t = (1/p)(p-1)(p/2)\, t = (p-1)\, t / 2 \qquad \square$$

Assumption 1: For simplicity we assume without loss of generality that $t = 1$.

The aim of patients is to minimize their (expected) waiting time. However, since in the representation of a game it is usual to understand that the players have the aim to maximize their own payoff, we define below the payoffs table of *G* by assuming that payoffs are given by the respective (expected) waiting times with "changed sign".

Patient I , Patient II	M	T
M	–1/2 , –1/2	0 , 0
T	0 , 0	– 1/2 , – 1/2

For example, if Patients I chooses M and Patient II choses M, then: Patient I will find at the doctor 2 persons (including himself), then the (expected) waiting time of Patient I is equal to 1/2 [according to formula (0)], and then his payoff is equal to –1/2. Let us observe that game *G* has multiple NE's in pure strategies, i.e., game *G* enjoys Case 3 of Chapter 0. Then, according to Chapter 0, let us try below to apply the EC Scheme.

Step 1: Calculation of EN of *G*.

Patient I , Patient II	M	T
M	–1/2 ,–1/2	<u>0</u> ,<u>0</u>
T	<u>0</u> ,<u>0</u>	–1/2 ,–1/2

Game *G* has 2 NE's in pure strategies and 1 EN in mixed strategies (different to pure) as follows:

- The first NE is (M, T) that is ((1, 0), (0, 1)) in terms of mixed strategy. This EN induces the probability distribution on $S_1 \times S_2$ which is represented in the following bi-matrix.

Patient I , Patient II	M	T
M	0	1
T	0	0

Then the expected payoff of each player is equal to **0**.
That can be deduced by considering both the bi-matrix of payoffs and the bi-matrix of probability distribution on $S_1 \times S_2$, and carry out the corresponding multiplications.
For example for Patient I one has: – (1/2) · 0 + (0) · 1 + (0) · 0 + – (1/2) · 0 = **0**

- The second NE is (T, M) that is ((0, 1), (1, 0)) in terms of mixed strategy. This EN induces the probability distribution on $S_1 \times S_2$ which is represented in the following bi-matrix.

Patient I , Patient II	L	M
L	0	0
M	1	0

Then the expected payoff of each player is equal to **0**.
That can be deduced by considering both the bi-matrix of payoffs and the bi-matrix of probability distribution on $S_1 \times S_2$, and carry out the corresponding multiplications.
For example for Patient I one has: – (1/2) · 0 + (0) · 0 + (0) · 1 + – (1/2) · 0 = **0**

- The third NE is ((1/2, 1/2), (1/2, 1/2) in mixed strategies (different from pure). This NE induces the probability distribution on $S_1 \times S_2$ represented in the following bi-matrix.

Patient I , Patient II	M	T
M	1/4	1/4
T	1/4	1/4

Then the expected payoff of each player is equal to **–1/4**.
That can be deduced by considering both the bi-matrix of payoffs and the bi-matrix of probability distributions on $S_1 \times S_2$, and carry out the corresponding multiplications.
For example for the Patient I one has: – (1/2) · 1/4 + (0) · 1/4 + (0) · 1/4 – (1/2) · 1/4 = **–1/4**

NOTE: The third NE in mixed strategies is, in practice, the behavior adopted by players.

Step 2: The two players nominate a Mediator which: (i) randomly draw an outcome of *G* on the basis of a probability distribution *p* on $S_1 \times S_2$ [that is the set of possible outcomes of *G*] which is a Correlated Equilibrium of *G* (possibly proposed by the players); (ii) recommends to each player a strategy on the basis of the drawn outcome of *G*.

By definition of Correlated Equilibrium, with reference to the table below, each probability distribution $p = (p_1, p_2, p_3, p_4)$ on $S_1 \times S_2$ = {(M, M), (M, T), (T, M), (T, T)} such that $p_1 = p_4 = 0$, $p_2 \geq 0$, $p_3 \geq 0$ is a Correlated Equilibrium of *G* [as the NE's in pure strategies of *G* are (M, T) and (T, M)].

Patient I , Patient II	M	T
M	p_1	p_2
T	p_3	p_4

For example *p* = (0, 3/5 , 2/5 , 0) is a Correlated Equilibrium of *G*.

In the case under consideration, the Mediator may be represented by the doctor, who will randomly draw either (M, T) or (T, M) and will recommend to each patient a different day (between M and T) on the basis of the drawn outcome.

Then two different scenarios for the game emerge: the *current scenario* and the *proposed scenario*.

The *current scenario* is represented by the EN in mixed (different from pure) strategies of *G*, i.e. ((1/2, 1/2), (1/2, 1/2)), and by the consequent expected payoff pair **(–1/4, –1/4)**: in this scenario every patient goes to the doctor by randomly choosing either Monday or Tuesday on the basis of the probability distribution (1/2, 1/2).

The *proposed scenario* is represented by a Correlated Equilibrium of *G*, that generates either outcome (M, T) or outcome (M, T), and by the consequent expected payoff pair **(0, 0)**: in this scenario every patient goes to the doctor on the basis of the recommendations provided by the doctor himself, who will divide the group of patients into two subgroups of equal size (in this case each group is formed by 1 patient) and will recommend to each subgroup a different day (between M and T).

Observation 1
In conclusion let us observe that the expected patients' payoffs in the proposed scenario are greater than those in the current scenario. □

1.2.2 The case with 2n patients and with 2 days

In this section let us try to extend section 3.2.1 to the case with $2n$ patients and with 2 days. Let us represent below an instance of this case by the following game *G* static with complete information.

Game *G*:

Players: {Patient 1, ... , Patient $2n$}

Strategies for each player: {L, M}

L = go to the doctor on Monday,

M = go to the doctor on Tuesday.

Payoffs:
for each player $i \in \{1, \dots, 2n\}$, the payoff $u_i(s_1, \dots, s_{i-1}, s_i, s_{i+1}, \dots, s_{2n})$ is equal to :
:: if s_i = L, then $u_i(s_1, \dots, s_{i-1}, s_i, s_{i+1}, \dots, s_{2n}) = (p_L - 1)/2$
where p_L is the number of strategies $s \in \{s_1, \dots, s_{i-1}, s_i, s_{i+1}, \dots, s_{2n}\}$ such that s = L
:: if s_i = M, then $u_i(s_1, \dots, s_{i-1}, s_i, s_{i+1}, \dots, s_{2n}) = (p_M - 1)/2$
where p_M is the number of strategies $s \in \{s_1, \dots, s_{i-1}, s_i, s_{i+1}, \dots, s_{2n}\}$ such that s = M

In this case the table of payoffs cannot be represented. Then the calculation of NE's in pure strategies of *G* and the calculation of NE's in mixed strategies (different from pure) of *G* will be done in a different way.

Step 1: Calculation of NE's of *G*.

Proposition 1

Let G be the game defined above. Then:

i) there exist $\binom{2n}{n}$ NE's in pure strategies of G: they are [all and only] the outcomes of G in which n patients choose M and n patients choose T;
(ii) there exists exactly 1 NE in mixed strategies (different from pure) of G: it is the outcome of G in which each patient chooses the mixed strategy (1/2, 1/2).

Proof. As a preliminary let us observe that: every NE of G is an outcome such that the (expected) number of patients who will go to the doctor on Monday is equal to the (expected) number of patients who will go to the doctor on Tuesday. This is since otherwise there would be at least one patient who has the incentive to deviate from his strategy.

Proof of (i): The assertion (i) is a direct consequence of the preliminary.

Proof of (ii): As a first step, let us show that the outcome of G in which each patient chooses the mixed strategy (1/2, 1/2), i.e., the outcome ((1/2, 1/2), ..., (1/2, 1/2)), is a NE in mixed strategies (different from pure) of G. This follows from the fact that, for any fixed patient i, for $i \in \{1, \ldots, 2n\}$, if all the other patients choose the mixed strategy (1/2, 1/2) [and this implies that the patient i will find at the doctor the same (expected) number of patients both on Monday and on Tuesday], patient i gets the same payoff for any strategy he chooses: in particular the mixed strategy (1/2, 1/2) is an optimal response to the strategies of all the other patients.
As a second step, let us show that there exists no other EN in mixed strategies (different from pure) of G. By contradiction let us assume that there exists another EN in mixed strategies (different from pure) of G. Then there exists a patient i, for $i \in \{1, \ldots, 2n\}$, which chooses a mixed strategy different from (1/2, 1/2) of G, say for example the mixed strategy (1/3, 2/3). Then patient i will get the same payoff, both by choosing M with probability 1/3 and by choosing T with probability 2/3. On the other hand by the preliminary, with respect to $2n$ patients, the (expected) number of patients that [according to this NE] will go to the doctor on Monday is equal to the (expected) number of patients that [according to this NE] will go to the doctor on Tuesday. It follows that, with reference to the other $2n - 1$ patients, the (expected) number of patients that [according to this NE] will go to the doctor on Monday is strictly greater than the (expected) number of patients that [according to this NE] will go to the doctor on Tuesday. Then the optimal response of patient i to the strategies chosen by all the other patients [according to this NE] is not (1/3, 2/3) but is (0, 1), a contradiction. □

NOTE: The NE in mixed strategies of (ii) is, in practice, the behavior adopted by players.

Step 2: The $2n$ players nominate a Mediator which: (i) randomly draw an outcome of G on the basis of a probability distribution p on $S_1 \times S_2 \times \ldots \times S_{2n}$ [that is the set of possible outcomes of G], which is a Correlated Equilibrium of G (possibly proposed by the players); (ii) recommends to each player a strategy on the basis of the drawn outcome of G.
For convenience we write $d = |S_1| \cdot |S_2| \cdot \ldots \cdot |S_{2n}|$. Then a probability distribution p on $S_1 \times S_2 \times \ldots \times S_{2n}$ may be written as $p = (p_1, p_2, \ldots, p_d)$.
By definition of Correlated Equilibrium: each probability distribution $p = (p_1, p_2, \ldots, p_d)$ on $S_1 \times S_2 \times \ldots \times S_{2n}$ such that
$p_i > 0$ for outcomes i in $S_1 \times S_2 \times \ldots \times S_{2n}$ which are NE's in pure strategies of G,
$p_i = 0$ for outcomes i in $S_1 \times S_2 \times \ldots \times S_{2n}$ which are not NE's in pure strategies of G,

is a Correlated Equilibrium of *G*.

In the case under consideration, the Mediator may be represented by the doctor that will randomly draw either (M, T) or (T, M) and will recommend to each patient a different day (between L and M) on the basis of the drawn outcome.

Then two different scenarios for the game emerge: the ***current scenario*** and the ***proposed scenario***.

The ***current scenario*** is represented by the NE in mixed strategies (different from pure) of *G*, which as remarked above corresponds to the behavior adopted by patients: in this scenario every patient goes to the doctor by randomly choosing either Monday or Tuesday on the basis of the probability distribution (1/2, 1/2).
The expected payoff of each patient, in such a scenario, is equal to:

$$-\frac{1}{2^{2n-1}}\sum_{k=0}^{n-1}\binom{2n-1}{k}\frac{2n-1}{2}$$

as shown later.

The ***proposed scenario*** is represented by a Correlated Equilibrium of *G*: in this scenario every patient goes to the doctor on the basis of the recommendations of the doctor himself, which will divide the group of patients into two subgroups of equal size (in this case each subgroup is formed by *n* patients) and will recommend to each subgroup a different day (between M and T).
The expected payoff of each patient, in such a scenario, is equal to

$$-(n-1)/2.$$

That follows by the (expected) waiting time formula (0), since *n* patient will go to the doctor on Monday and *n* patient will go to the doctor on Tuesday.

Let us introduce below a parenthesis and two propositions in order to compare the expected payoffs of patients in the current scenario and in the proposed scenario.

Parenthesis

In this parenthesis let us report some known equations.

(1) $\forall n, k, n \geq k$, one has: $\binom{n}{k} = \binom{n}{n-k}$
$\binom{n}{k}$ is the number of ways in which one can choose *k* objects in a set of *n* objects.

(2) $\forall n$ one has: $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}$

(3) $\forall n, k, n \geq k$, one has: $\sum_{k=0}^{2n-1}\binom{2n-1}{k}\frac{k}{2} = \sum_{k=0}^{n-1}\binom{2n-1}{k}\frac{2n-1}{2}$.

Concerning equations (1) e (2), they are well-known, so that their proof is omitted.

Concerning equation (3), let us report below just an intuitive proof.

For example, assume that n = 4, and then that $2n$ = 8.

The right-side sum of the (3) is equal to:

$$\binom{7}{0}0 + \binom{7}{1}\frac{1}{2} + \binom{7}{2}\frac{2}{2} + \binom{7}{3}\frac{3}{2} + \binom{7}{4}\frac{4}{2} + \binom{7}{5}\frac{5}{2} + \binom{7}{6}\frac{6}{2} + \binom{7}{7}\frac{7}{2}$$

$$= \binom{7}{0}0 + \left[\binom{7}{1}\frac{1}{2} + \binom{7}{6}\frac{6}{2}\right] + \left[\binom{7}{2}\frac{2}{2} + \binom{7}{5}\frac{5}{2}\right] + \left[\binom{7}{3}\frac{3}{2} + \binom{7}{4}\frac{4}{2}\right] + \binom{7}{7}\frac{7}{2}$$

By equation (1) the left-side sum of equation (3) can be obtained as follows:

$$= \binom{7}{1}\frac{7}{2} + \binom{7}{2}\frac{7}{2} + \binom{7}{3}\frac{7}{2} + \binom{7}{7}\frac{7}{2} = \binom{7}{0}\frac{7}{2} + \binom{7}{1}\frac{7}{2} + \binom{7}{2}\frac{7}{2} + \binom{7}{3}\frac{7}{2}$$

Then equation (3) is proved. □

Proposition 2

The expected payoff of each patient in the *current scenario* is equal to:

$$-\frac{1}{2^{2n-1}}\sum_{k=0}^{n-1}\binom{2n-1}{k}\frac{2n-1}{2}$$

Proof. Without loss of generality, we calculate the (expected) waiting time [which is equal to the expected payoff with changed sign] of a patient X. Remember that patient X chooses the mixed strategy (1/2, 1/2) [and that all the other patients choose the mixed strategy (1/2, 1/2)]. Then the (expected) waiting time of patient X is given by 1/2 E(L) + 1/2 E(M) where E(M) and E(T) are respectively the (expected) waiting times of patient X by choosing M and by choosing T. Then, since by symmetry E(M) = E(T), the (expected) waiting time of patient X is equal to E(M) without loss of generality.

Then let us calculate E(M). Then let us assume that patient X chooses M and that all the other patients choose the mixed strategy (1/2, 1/2).

The number of ways/cases for which k patients (not included X) will go to the doctor on Monday is equal to $\binom{2n-1}{k}$. Note that, for each of such cases, the expected waiting time of patient X is:

$$\frac{(k+1)-1}{2} = \frac{k}{2}$$

That follows by the (expected) waiting time formula (0), since k + 1 patients (included X) will go to the doctor on Monday.

It follows that

$$\mathrm{E(M)} = \frac{1}{2^{2n-1}}\sum_{k=0}^{2n-1}\binom{2n-1}{k}\frac{k}{2}$$

since $\frac{1}{2^{2n-1}}$ is the probability that each of such cases occurs.

Then by equation (3) one has:

$$\frac{1}{2^{2n-1}}\sum_{k=0}^{2n-1}\binom{2n-1}{k}\frac{k}{2} = \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1}\binom{2n-1}{k}\frac{2n-1}{2}.$$

Then the proposition is proved. □

Proposition 3

The expected payoff of a patient in the *proposed scenario* minus the expected payoff of a patient in the *current scenario* is equal to 1/4.

Proof. For simplicity, let us consider the expected waiting times, that is the expected payoffs with changed sign. As a preliminary let us observe that:

$$\frac{1}{2^{2n-1}}\sum_{k=0}^{n-1}\binom{2n-1}{k} = \frac{1}{2^{2n-1}}\frac{1}{2}\sum_{k=0}^{2n-1}\binom{2n-1}{k} = \frac{1}{2}\ .$$

In fact:

$\sum_{k=0}^{n-1}\binom{2n-1}{k} = \frac{1}{2}\sum_{k=0}^{2n-1}\binom{2n-1}{k}$ by equation (1),

$\frac{1}{2^{2n-1}}\sum_{k=0}^{2n-1}\binom{2n-1}{k} = 1$ by equation (2).

Then, by the above and by Proposition 2, the expected waiting time of a patient in the current scenario is:

$$\frac{1}{2}\frac{2n-1}{2} = \frac{2n-1}{4}.$$

Then the expected waiting time of a patient in the current scenario minus the expected waiting time of a patient in the proposed scenarios is:

$$\frac{2n-1}{4} - \frac{n-1}{2} = \frac{2n-1-2(n-1)}{4} = \frac{1}{4}$$ □

Observation 2

In conclusion let us observe that the expected patients' payoffs in the scenario proposed are greater than those in the current scenario. □

Observation 3

Let us observe that the "advantage" which a patient gets in the proposed scenario with respect to the current scenario:

i) is equal to (1/4) t , where t is the average time for a visit, according to Assumption 1;

ii) does not depend on n, i.e., does not depend on the number of patients. □

1.2.3 The general case

The general case is that with *n* patients and with *m* days. Let *G* be the corresponding game. We did not study this case. However, by assuming for simplicity that

$$n = k \cdot m$$

for a natural *k*, it seems to be natural to conjecture the following facts:

R1) The NE's in pure strategies of *G* are [all and only] the outcomes of *G* in which the group of patients is divided into *m* subgroups of cardinality *k*, and each different subgroup of patients chooses a different day.

R2) there is exactly 1 NE in mixed strategies (different from pure) of *G*: it is the outcome of *G* in which each patient chooses the mixed strategy (1/*m*, …, 1/*m*), i.e., each patient goes to the doctor by randomly choosing a day on the basis of the probability distribution (1/*m*, …, 1/*m*) on the days set [which is formed by *m* days].

Then two different scenarios for the game emerge: the ***current scenario*** and the ***proposed scenario***.

The ***current scenario*** is represented by the NE in mixed strategies (different from pure) of *G*, which as specified above corresponds to the behavior adopted by patients: in this scenario every patient goes to the doctor by randomly choosing a day on the basis of the probability distribution (1/*m*, …, 1/*m*) on the days set [which is formed by *m* days].

The ***proposed scenario*** is represented by a Correlated Equilibrium of *G*: in this scenario every patient goes to the doctor on the basis of the recommendations provided by the doctor himself, which will divide the group of patients into *m* subgroups of cardinality *k* and will recommend to each different subgroup a different day.

R3) The expected patients' payoffs in the scenario proposed are greater than those in the current scenario: in particular the "advantage" which a patient gets in the proposed scenario with respect to the current scenario does not depend on *n*, i.e., does not depend on the number of patients.

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Chapter 2: Tax evasion

The main references in this chapter are those of Chapter 0, a work of Alessio Monticelli [9], and the graduate theses of Matteo Galliani [8] and of Ileana Staniscia [12].

The problem of tax evasion is treated in the literature by several authors as a Game Theory problem.

A usual approach is to consider [in the minimal case] a game *G* with two players (cf. [9, 10, 11]): one is the Taxpayer (i.e. the Citizen) and one is the Tax Authorities (i.e., the State); the Taxpayer can choose either to evade taxes or to pay taxes; the Tax Authorities can choose either to check the Taxpayer or not to check the Taxpayer [9].

An alternative approach, based on the fact that "we are the State", may be to consider [in the minimal case] a game *G* with two players: one is a Taxpayer and one is another Taxpayer; each Taxpayer can choose either to evade taxes or to pay taxes [8, 10].

2.1 A usual approach: Taxpayer vs Tax Authority

The problem of tax evasion is described by focusing on the relationship between Taxpayer and Tax Authority by the following game *G* static with complete information:

Game *G*:

Players : {C , F}

Strategies : SC = {E, P}, SF = {A , NA}

where:

C is the Taxpayer,

F is the Tax Authority,

SC is the set of strategies of C: he can choose either to pay taxes (P) or to evade taxes (P);

SF is the set of strategies of F: he can choose either to ascertain (A) or not to ascertain (NA) [whether C has paid or evaded].

In order to define the *Payoffs* for each combination of strategies we consider the following data.

We assume that :

:: the tax amount is equal to 10;

:: the penalty is 100% : then the penalty represents a payoff of –10 for the Taxpayer and a payoff of 10 for the Tax Authority;

:: the cost to ascertain is 3% of the tax amount for F and 10% of the tax amount for C: then the ascertain represents (in any case) a payoff of –0,3 for F and a payoff of –1 for C.

Then let us build the table of game *G*.

Tax Authority F

		A	NA
Taxpayer C	P	− 1 ,− 0,3	0 , 0
	E	−11 , 9,7	10 , − 10

In game *G* (with these payoff):

:: there is no NE in pure strategies,

:: there is 1 NE in mixed strategies : ((P, E), (A, NA)) = ((0.985, 0.015), (0.5, 0.5)).

2.2 An alternative approach: Taxpayer vs Taxpayer

The problem of tax evasion is described on the basis of the consideration that "we are the State", i.e. by focusing on the relationship between two Taxpayers [as a minimal case], by the following game *G* static with complete information:

Game *G*:

Players : {C1 ; C2}

Strategies : S1 = {P ; E} , S2 = {P ; E}

where:

C1, C2 are the two Taxpayers who make up the Community;

S1 is the set of strategies of C1: he can choose either to pay taxes (P) or to evade taxes (E);

S2 is the set of strategies of C2: he can choose either to pay taxes (P) or to evade taxes (E).

In this context the taxes must be understood as the advantage that each Taxpayer provides to the Community and to himself as a part of the Community. If a Taxpayer decides to evade taxes, then he harms the Community for that missed contribution. The situation can therefore be displayed using the following parameters:

• V = advantage that every Taxpayer in the Community has when any Taxpayer chooses P.

• T = expected amount that every Taxpayer gains by choosing E, i.e.,

$$T = I - u$$

where I is the tax amount and u is the expected penalty (which relies both on the penalty and on the probability that Tax Authority carries out the check).

Then let us build the table of the game *G*.

		Taxpayer C2	
		P	E
Taxpayer C1	P	$2V , 2V$	$V , V + T$
	E	$V + T, V$	T , T

A comment on how the *Payoffs* are formed:

• If C1 chooses P, then we have two possible occurrences:

if C2 chooses P, then we have the strategies pair (P ; P): that is C1 and C2 regularly pay taxes, so that for example C1 obtain V thanks to C1 (himself) plus V thanks to C2; the result is the payoffs pair ($2V$, $2V$);

if C2 chooses E, then we have the strategies pair (P ; E): that is C1 pays taxes, while C2 evades taxes; thus they both obtain V thanks to C1, but C2 obtains in addition T, i.e., the amount that C2 gains by the evasion; the result is the payoffs pair (V , $V+ T$).

• If C1 chooses E, then we have two other possible occurrences:

if C2 chooses P, then we have the strategies pair (E ; P): that is the situation is specular to that above; the result is the payoffs pair ($V +T$; V);

if C2 chooses E, then we have the strategies pair (E ; E): that is C1 and C2 evade taxes and then they both get T; the result is the payoffs pair (T ; T).

Let us observe that:

:: If C1 chooses P, then P is an optimal response of C2 if and only if $V \geq T$.

:: If C1 chooses E, then P is an optimal response of C2 if and only if $V \geq T$.

:: If $V > T$, then P is a strictly dominant strategy for both C1 and C2, i.e., for each Taxpayer strategy P is an optimum response for any strategy of the other Taxpayer;

:: If $V < T$, then E is a strictly dominant strategy for both C1 and C2, i.e., for each Taxpayer strategy P is an optimum response for any strategy of the other Taxpayer.

Then we can conclude that:

Proposition 1

The following statements hold:

(i) if $V > T$, then G has a unique NE, i.e., (P ; P);

(ii) if $V < T$, then G has a unique NE, i.e., (E ; E);

(iii) if $V = T$, then G has four NE, i.e., each pair of strategies. □

Let us try to comment this proposition by the following two observations.

Observation 1

By Proposition 1, (P ; P) is a NE of G if and only if $V \geq T$ and then if and only if

$$V \geq I - u.$$

Let us make explicit the formula of u:

$u = p\,(I + S) + 0\,(1 - p) = p\,(I + S)$,

where S is the penalty and p is the probability that Tax Authority carries out the check.

Then (P ; P) is induced to be a NE of G, namely the Taxpayers are induced to choose to pay taxes, as much as:

• the value of V increases: this is achieved through an efficient use (however advantageous for Taxpayers) by the State of the money obtained by taxes;

• the value of I decreases: this is achieved through a lower tax pressure by the State;

• the value of S increases: this is achieved by increasing the penalty by the State;

• the value of p increases: this is achieved by enhancing the check by the State. □

Observation 2

If one has $T/2 \leq V < T$, then a paradox occurs: by Proposition 1, the NE [i.e., the predicted outcome] of G is (E ; E), though the outcome (P ; P) ensures to both players a strictly greater payoff.

For example if:

$V = 7$

$T = 10$

then one has the following table of the game.

		Taxpayer $C2$	
		P	E
Taxpayer $C1$	P	14 , 14	7 , <u>17</u>
	E	<u>17</u> , 7	<u>10</u> , <u>10</u>

The pair of strategies (E ; E) is a NE with associated payoffs (10 , 10). On the other hand there is the pair of strategies (P ; P) with associated payoffs (14, 14) that for Taxpayers are strictly better than payoffs (10, 10). Then each Taxpayer prefers to evade taxes though each Taxpayer would get a strictly grater payoff if they both pay taxes. □

2.2.1 A "way out"

This kind of paradox was already observed in other contexts, such as in the famous *Prisoner's Dilemma* game, and is based on the mutual lack of confidence between the players. In particular for convenience let us formalize this kind of paradox by the following definition.

Definition 1 (Critical Situation)

A *critical situation* is a game G static with complete information that admits both an outcome which is a NE of G with associated payoff $(e_1,\ldots, e_n)$, which we shall call as *Nash outcome*, and an outcome which is not a NE of G with associated payoff $(x_1,\ldots, x_n)$, such that $x_i > e_i$ for $i = 1, ..., n$, which we shall call as *collusive outcome*. □

Then the "tax evasion" game, i.e., the game G defined in Section 4.2, may be a critical situation for certain values of payoffs as shown in Observation 2. There are several real-life situations in social dynamics which may be described as critical situations – a well-known example is that of the *Arms Race* game – thus with negative consequences for society. Many scientists have sought a way out of these critical situations.

It is known that a *way out* from a critical situation, say a game G, is to infinitely repeat the game G in order to apply the Friedman's Theorem, see Gibbons [2], see also Aumann [7].

Then below we try to apply this *way out* with respect to the "tax evasion" game.

To this end we quote some notions from Gibbons [2].

Infinitely repeated games

Infinitely repeated games are dynamic games with complete information.

Definition 2 (Infinitely repeated games)

Given a game G static with complete information, called *constituent game*, given an interest rate r, given a probability p, and given then the discount factor $\delta = (1-p) / (1+r)$, let $G(\infty,\delta)$ denote the infinitely repeated game that consists in playing in an infinite number of stages the game G, i.e., for $t = 1, 2, \ldots$ at stage t-th the game G is played. For $t = 1, 2, \ldots$, the outcomes of the previous $t-1$ stages of the constituent game are known before the t-th stage begins. The payoff of each player in $G(\infty,\delta)$ is the current value of the payoff that the player obtains by the infinite sequence of constituent games. □

Let us report the definition of the values r, p, δ, and current value.

These values are introduced since in a game repeated infinitely the simple sum of the payoff arising from the infinite sequence of constituent games does not represent an adequate measure of the payoff. For example, getting a payoff of 4 in each period is better than getting a payoff of 1 in each period, but the infinite sum of the payoffs produces as a result the value "infinite" in both cases. It is also introduced the probability that the game terminates at any stage.

Interest rate r

Let r be the interest rate applicable to each stage of the game.

Probability p

Let p be the probability that at any stage the game terminates [and then let $1 - p$ be the probability that at any stage the game continues].

Discount factor δ

Let $\delta = (1 - p) / (1 - r)$ be the discount factor which represents both the current value of a dollar that will be received in the next stage, and the possibility that the game will terminate in the next stage. Note that by definition one has

$$0 < \delta < 1.$$

Current value

Given the discount factor δ and given an infinite sequence of payoffs$\pi_1, \pi_2, \pi_3, \ldots$.[associated to the infinite sequence of constituting games] the current value of $\pi_1, \pi_2, \pi_3, \ldots$ is

$$\pi_1 + \delta\, \pi_2 + \delta^2\, \pi_3 + \ldots$$

Predicted outcome of infinitely repeated games

According to Game Theory the predicted outcome of an infinitely repeated game $G(\infty,\delta)$ [as a dynamic game with complete information] is that generated by the Subgame Perfect Nash Equilibria of $G(\infty,\delta)$ introduced by Selten (1965) [2].

Definition 3 (Subgame Perfect Nash Equilibrium)

A Subgame Perfect Nash Equilibrium of $G(\infty,\delta)$ is a Nash Equilibrium of $G(\infty,\delta)$ that remains a Nash Equilibrium for every subgame of $G(\infty,\delta)$. □

Friedman's Theorem

We quote Friedman's Theorem with reference only to critical situations.

Theorem 1 (Friedman, 1971)

Let G be a finite game, static with complete information. Let us assume that G is a critical situation: let $(e_1,\ldots,e_n)$ be the payoffs vector of associated to a Nash outcome of G and let $(x_1,\ldots, x_n)$ be the payoffs vector associated to a collusive outcome of G. If δ is sufficiently close to 1, then there is a

Subgame Perfect Nash Equilibrium of the infinitely repeated game $G(\infty,\delta)$ that generates an outcome where in each stage the players choose for the collusive outcome of G. □

For example consider the “tax evasion” game G with the payoffs of Observation 2:

		Taxpayer $C2$	
		P	E
Taxpayer $C1$	P	14 , 14	7 , <u>17</u>
	E	<u>17</u> , 7	<u>10</u> , <u>10</u>

One has that:

$(e_{1,}e_2) = (10, 10)$ Nash outcome,

$(x_{1,}x_2) = (14, 14)$ collusive outcome.

Let us report below an outline of Friedman’s Theorem proof, with reference to the infinitely repeated game $G(\infty,\delta)$, with constituting game G of Observation 2.

The outline can be summarized in the following two steps.

<u>How to obtain such a Subgame Prefect Nash Equilibrium of $G(\infty, \delta)$</u>

Such a Subgame Prefect Nash Equilibrium is obtained by introducing *trigger strategies*.

An example of a trigger strategiesis given by the two strategies below, namely TS1 and TS2 which are respectively strategies of C1 and of C2.

TS1: In the first stage of $G(\infty, \delta)$, choose P. In the *t*-th stage of $G(\infty, \delta)$, if the outcome of all previous stages was (P, P), then choose P, otherwise choose E.

TS2: In the first stage of $G(\infty, \delta)$, choose P. In the t-th stage of $G(\infty, \delta)$, if the outcome of all previous stages was (P, P), then choose P, otherwise choose E.

Strategy TS1 can be commented as follows: C1 proposes to C2 to choose for the collusive outcome of G in all stages of $G(\infty, \delta)$, with the threat that if C2 should not accept such a proposal (i.e. should “deviate”) in a certain stage of $G(\infty, \delta)$ then C1 would choose for the Nash outcome of G in all the successive stages of $G(\infty, \delta)$.

Strategy TS2 can be commented similarly by symmetry.

Then, if both players adopt such trigger strategies, then the outcome of $G(\infty, \delta)$ is that in each stage of $G(\infty, \delta)$ players choose for the collusive outcome of G [i.e. for (P, P)].

<u>How to calculate the values of δ sufficiently close to 1</u>

To complete Friedman’s Theorem proof let us show that:

(TS1, TS2) a Subgame Perfect Nash Equilibrium of $G(\infty,\delta)$ provided that δ is sufficiently close to 1.

To this end, since each subgame of an infinitely repeated game is identical to the original game, it is sufficient to prove that (TS1, TS2) is a Nash Equilibrium of $G(\infty,\delta)$ (i.e., TS1 and TS2 form a pair of strategies that are mutually optimal responses) provided that δ is sufficiently close to 1.

By symmetry it is sufficient to proceed as follows.

Let us assume that C1 adopts TS1, and let us show that TS2 is an optimal response of C2 to TS1, provided that δ is sufficiently close to 1.

Case 1: C2 adopts TS2

then C2 obtains the payoff:

$14+ \delta\cdot 14+ \delta^2\cdot 14+\ldots+ \delta^n\cdot 14+\ldots$

Case 2: C2 does not adopt TS2

If C2 chooses P at each stage of $G(\infty, \delta)$, then C2 obtains the same payoff as in Case 1.

Otherwise, C2 "deviates" in a certain stage k+1 and its payoff is less than or equal to

$3 + \delta\cdot 3 + \ldots + \delta^{k-1}\cdot 3 + \delta^k\cdot 4 + \delta^{k+1}\cdot 1 +\ldots+ \delta^n\cdot 1 + \ldots$

The difference between the payoff of C2 in Case 1 and in Case 2 is less than or equal to

$\delta^k [(14 + \delta\cdot 14 + \delta^2\cdot 14 + \ldots + \delta^n\cdot 14 + \ldots) - (17 + \delta\cdot 10 + \delta^2\cdot 10 + \ldots + \delta^n\cdot 10 + \ldots)]$

Since $0 <\delta< 1$ one has: $1 + \delta+ \delta^2 + \ldots+ \delta^n+ \ldots = \delta/ (1 - \delta)$

Then the above difference is less than or equal to

$\{14 / (1 -\delta)\} - \{17 + [(\delta/ (1 -\delta)] \cdot 10\}$

and is not negative if

$14 / (1 -\delta) \geq 17 + [(\delta/ (1 -\delta)] \cdot 10$

i.e., if

$\delta \geq 3/7$

Therefore, if $\delta \geq 3/7$, then for C2 there is convenience to "deviate".

Then (TS1, TS2) is a Subgame Perfect Nash Equilibrium of $G(\infty,\delta)$ provided that $\delta \geq 3/7$.

<u>Conclusion of the "Way Out": decision support</u>

The alternative approach – i.e. the game *G* of the present Section 4.2 – highlights the possibility of a paradox [for certain payoff] according to Observation 2: that is it is possible that each Taxpayer chooses [in the context of a Nash Equilibrium, i.e., reasonably] to evade taxes though each Taxpayer would get a greater payoff if both Taxpayers pay taxes. This paradox is similar to that of the well-known *Prisoner's Dilemma* game and is based on the mutual lack of confidence between the players.

However there is a "way out" for this paradox, which consists in repeating the game infinitely, i.e. in playing the game $G(\infty,\delta)$ where δ is a certain discount rate. That would ensure that in each stage of $G(\infty, \delta)$ Taxpayers will choose to pay taxes [in the context of a Nash Equilibrium, i.e., reasonably] according to Friedman's Theorem for values of δ sufficiently close to 1.

To implement the game $G(\infty,\delta)$ in *the real-life* the following conditions seem to be necessary:

Condition (a): the payment of taxes should be done periodically (i.e., in a repeated manner): in particular each taxes payment deadline corresponds to each stage of $G(\infty,\delta)$;

Condition (a) is already implemented in the real-life.

Condition (b): Taxpayers should know, before each stage of $G(\infty,\delta)$, the outcome of the previous stages of $G(\infty,\delta)$.

Condition (b) requires the following two conditions:

Condition (b)*1: Tax Authority should check for every Taxpayer, before each stage of $G(\infty, \delta)$ takes place [i.e. before each payment deadline of taxes], if in the previous stages of $G(\infty,\delta)$ [i.e. if in the previous deadlines] he has evaded the taxes or has paid the taxes.

Condition (b)*2: Tax Authority should make public the results of the check.

A comment on Condition (b)*1: it requires that this check should be carried out in a short time by Tax Authority (also in order to let a Taxpayer possibly remedy before the next stage); maybe to this purpose an easy modality of tax payment would be helpful.

A comment on Condition (b)*2: it requires that Tax authority should make public for every Taxpayer if he has evaded the taxes or has paid the taxes; however, Tax Authority can avoid to make public the tax amount that he had to pay, thus saving some privacy.

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